An overview of the work of János Pintz in Number Theory

(from the perspective of an analyst)

Szilárd Gy. Révész

Number Theory Conference
Debrecen, July 7, 2022

A very brief Cv

Pintz on Dirichlét's L functions

János as my advisor – and his influence on my work

The Heilbronn Conjecture

Prime differences

Landau problems

The behavior of the remainder term of the PNT and the ζ-zeros

Some personal memories from the 70's

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He joined the Institute of Mathematics (now Rényi Institute) in 1977, where he is still working as professor emeritus.

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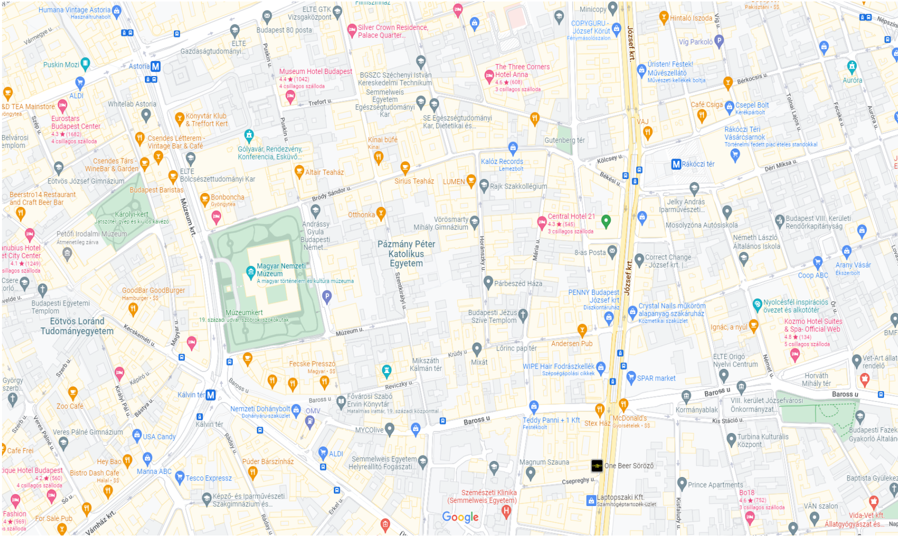
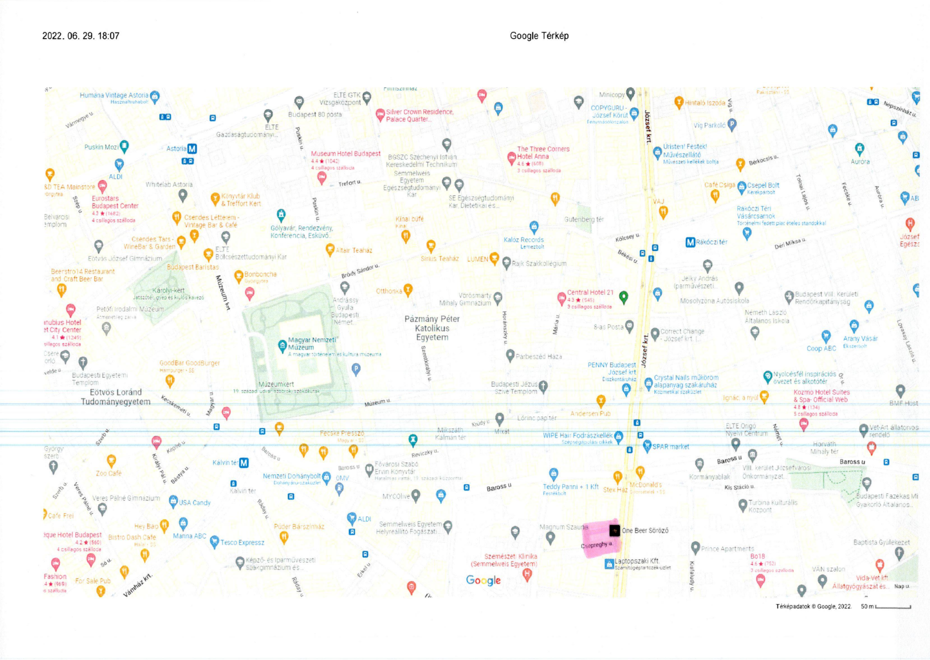
The "Nyócker", the VIIIth District of Budapest

2022. 06. 29. 18:07 Google Térkép

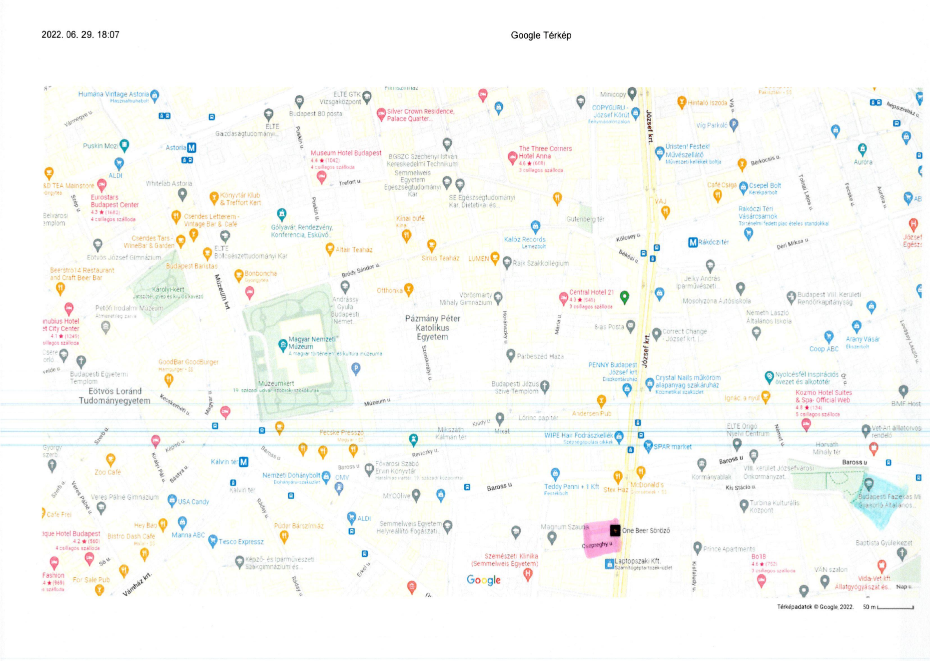


Térképadatok © Google, 2022. 50 m

Csepreghy street, where János lived till 1979



Fazekas Mihály Gimnázium



Eötvös University, Faculty of Sciences in the 70's

The (Alfréd Rényi) Institute of Mathematics

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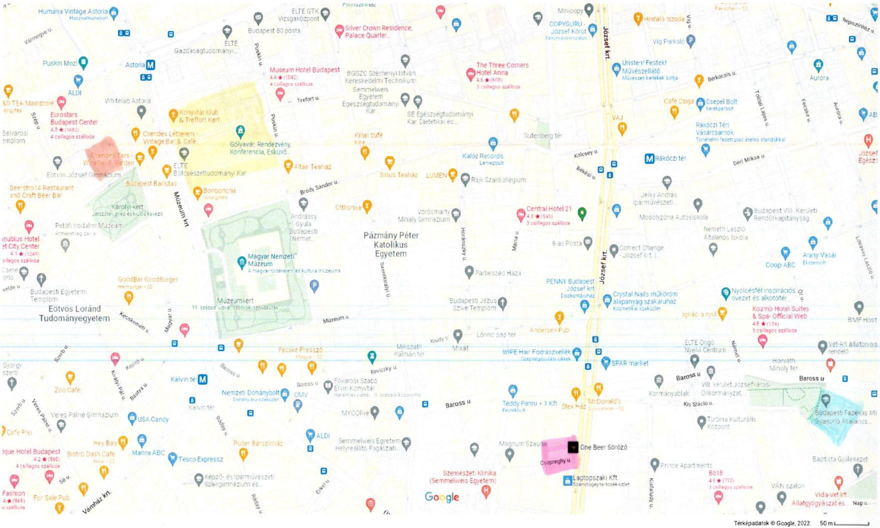
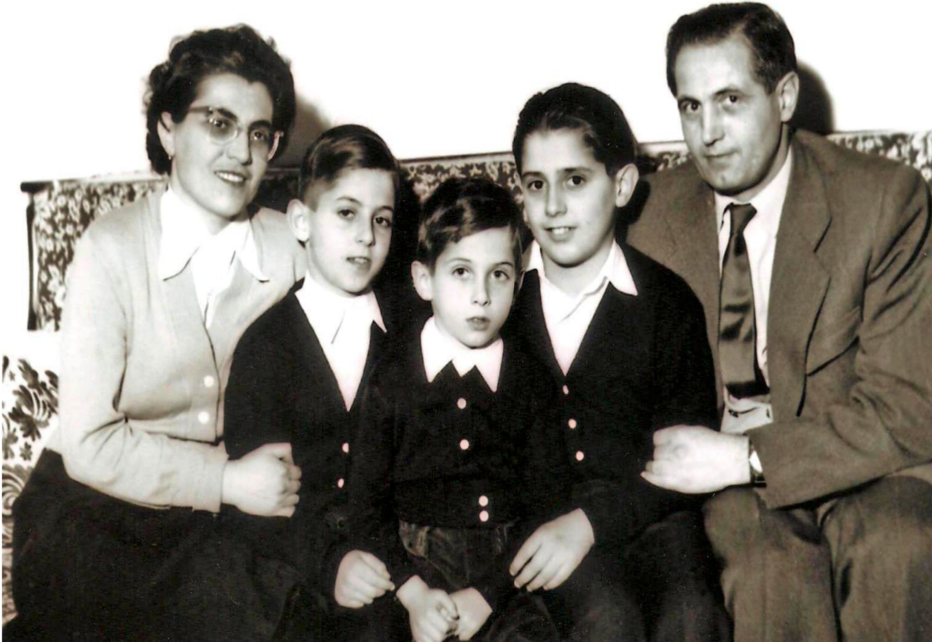


János with parents and brothers



János and Magdi on their wedding

János married with Magdi Berán in 1979

Daughters and grandchildren

They have two daughters and four grandchildren

Wikipedia list of main achievements

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- Goldston, S. W. Graham, Pintz, and Yıldırım proved that the difference between numbers which are products of exactly 2 primes is infinitely often at most 6.

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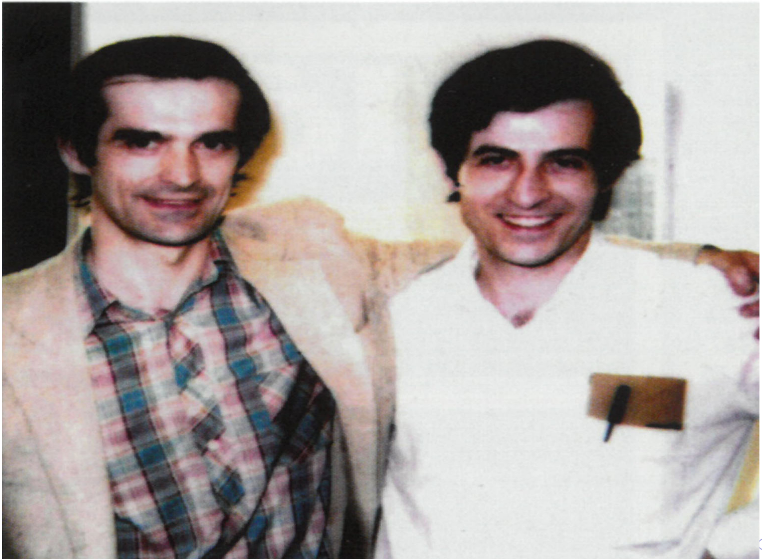
With his friend István Gyöngy

During his studies and later professional activities he established long-lasting close friendships, too.



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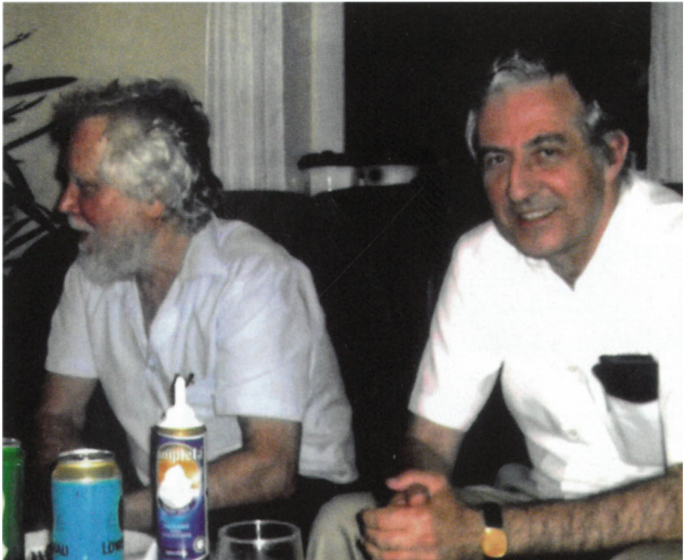
With another friend Endre Szemerédi

Apart from being an excellent mind, János is also a very good person, who makes friends easily.



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A photo shot from Yesterday

János and Kálmán are also good friends.

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With the jubilees - twelve years ago

Attila Pethő, Kálmán Győry,

Endre Sárközy, János Pintz

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- Rényi Alfréd Díj, Alfréd Rényi Institute of Mathematics (1974)

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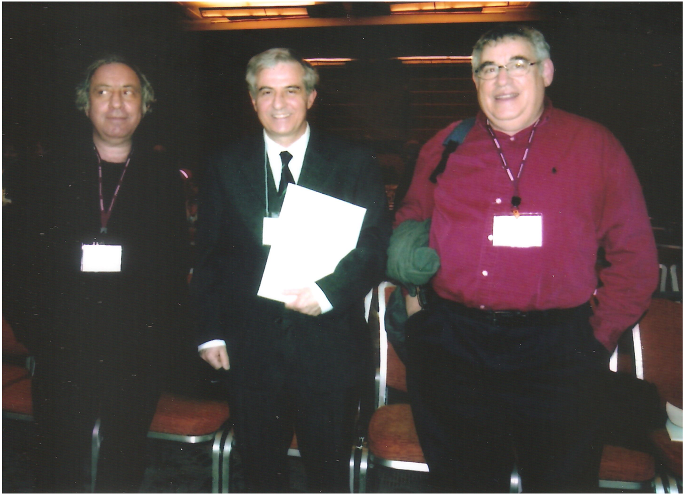
- Grünwald Géza emlékérem, Bolyai János Matematikai Társulat (1974)

Humboldt Fellowship, 1984-85

János at the time of his Humboldt Fellowship

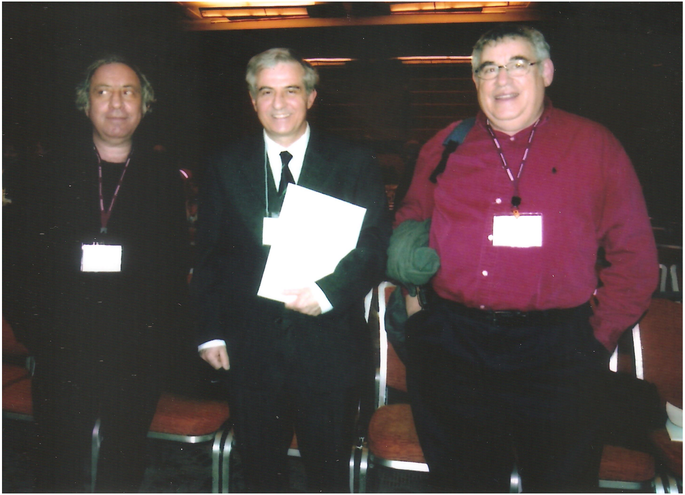
Receiving the Cole Prize

Yıldırım, Pintz and Goldston



Receiving the Széchényi Prize

With Endre Szemerédi, who also received another prize



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Pintz wrote about a dozen papers on Dirichlét's L functions[2] and their applications in which he reconsidered many classical results.

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[3] This was János' first paper, working out a question and following hints by Turán - this started him off in number theory already at the age of 20.

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Although he published his first two papers[4] in Hungarian, both were seriously refereed in the Mathematical Reviews.

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János' advisor, the late Pál Turán

János is still profoundly grateful to and has the kindest memories of Paul Turán, his early mentor and advisor, who attracted him to number theory and started him off in mathematics research.

A series on L functions

Then he wrote a series I-IX of similar title[5]. These studies went quite deep, with proving of density theorems, estimations for $L(1,\chi)$ and $L(s,\chi)$ values, Siegel zeros, analysing the Heilbronn and Deuring phenomenons, Landau-Page theorem, a new proof of a result of Wolke, and the Siegel-Val'fis theorem.

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One of the characteristics of his studies were that in many cases he found more direct, "quasi-elementary" proofs, with competitive, if not better end results.

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E.g. about the result on the Heilbronn phenomenon, Diamod writes: "The proof of this result is ingenious and quite brief, exploiting $g(n)$...".

Later, on the elementary proof of the Linnik-Vinogradov estimate $P(p) \ll p^{1/4+\varepsilon}$, where $P(p)$ denotes the least quadratic non-residue mod p, Diamond writes: "The present proof, like the original, involves Burgess' character sum estimates and Siegel's lower estimate of $L\left(1, \left(\frac{n}{p}\right)\right)$. However, the use of complex integration is avoided here by another application of the arithmetic function $g(n)$..."

[6] MR485731-MR485738

Density estimates for zeroes of L functions

In the IXth piece[7] of his series on L-functions, he gave an elegant application of *zero-detecting sums* and *large sieve inequalities* combined with ideas of Halász and Turán in proving the very sharp density estimates for L functions:

$$\sum_{q \leq Q} \sum_{\chi \mod q}^{*} N(\sigma, T, \chi) \ll Q^{c(1-\sigma)} T^{C(1-\sigma)^{3/2}} \log^K(QT),$$

the constants being $c = 16, C = 1600, K = 19$ in his proof.

[7] Elementary methods in the theory of L-functions. IX. Density theorems. *Acta Arith.* **49** (1988), no. 4, 387–394.

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Recently[8], János proved "log-free" density theorems, too. E.g.

$$\sum_{\chi \bmod q} N(\sigma,T,\chi) \ll (qT)^{\frac{(3+\varepsilon)(1-\sigma)}{8-6\sigma}}.$$

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János advising me on L functions and more

János also advised me in writing my very first paper[9] which, in reality, owes much more to him[10] than me. In it it was proved that in Dirichlét's Theorem the first prime $\equiv a \mod k$ occurs before $\exp(c\sqrt{k}\log^{11} k)$, a result weaker than Linnik's and Chen's, but quite good for only real-valued elementary methods.

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Altogether I wrote (or is still writing) some ten papers "under the influence" of his.

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His influence on my own research is particularly strong on the connection between zero distribution of ζ functions and oscillation order of the remainder term in the PNT.

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Decades after his "Elementary methods ... IX. Density theorems", recently[11] I followed his proof in getting zero density estimates[12] for Beurling zeta functions.

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If there are n points in the unit square, then there always exists three such that their triangle has area below c/n^2.

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Theorem (Komlós-Pintz-Szemerédi, 1981-82)

There exists a point configuration such that all triangles have area larger than $c \log n/n^2$.

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The problem of the true minimax here still stays (already for 40 years !) where they have left it (between $\log n/n^2$ and $n^{-8/7}$).

The twin prime conjecture

Twin Prime Problem / Conjecture: Are there ∞ many primes p with $p+2$ also prime?

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Best of XX$^{\text{th}}$ Century – Maier, 1988:

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log n} < 0.249.$$

Small gaps between primes

First breakthrough[13]: Goldston-Pintz-Yıldırım, 2005:

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log n} = 0,$$

i.e. $p_{n_k+1} - p_{n_k} = o(\log n_k)$ for some $n = n_k \to \infty$.

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Moreover, under the Elliott-Halberstam Conjecture on the level of distribution of primes in arithmetic progressions, they even obtained

$$\liminf_{n\to\infty} (p_{n+1} - p_n) < 16.$$

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$$\liminf_{n\to\infty} (p_{n+1}-p_n) < 16.$$

Similarly, if for any $\theta > 1/2$ θ-EH holds, we have

$$\liminf_{n\to\infty} (p_{n+1}-p_n) < C(\theta).$$

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Level of uniform distribution of primes

Definition

The level of uniform distribution of primes in arithmetical progressions is the supremum λ of all numbers θ with the following property: For all $A > 0$ and x large it holds

$$\sum_{q<x^\theta} \max_{a \mod q} \left| \sum_{p\le x, p\equiv a \mod q} \log p - \frac{x}{\varphi(q)} \right| < C_A \frac{x}{\log^A x}.$$

Results on the distribution level λ of primes

Theorem (Rényi, 1948)

$\lambda > 0$.

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Conjecture (θ-Elliott-Halberstam (θ-EH))

$\lambda \geq \theta$

Unfortunately, improving the Bombieri-Vinogradov Theorem resisted all efforts for half a century.

Further analysis of the method of GPY

Given the strength of the method, the natural question arises: what is the limit of the method? Can we reach e.g. the quasi-twin prime (i.e. "bounded gaps") conjecture?

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Key to the method is a certain sieving technique, based on a weighted Selberg sieve with cleverly constructed Selberg-coefficients

$$\lambda_d := \mu(d) P\left(\frac{\log(R/d)}{\log R}\right), \quad \left(R = X^{\alpha},\ 0 < \alpha < 1/2,\ X \approx p\right).$$

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$$\lambda_d := \mu(d) P\left(\frac{\log(R/d)}{\log R}\right), \quad \left(R = X^{\alpha},\ 0 < \alpha < 1/2,\ X \approx p\right).$$

where the necessary conditions for the weight function P are:

- $P(1) = 1$;
- $P \in C^{k-1}[0,1],\ P^{(k-1)} \in AC[0,1]$;
- $P(0) = P'(0) = \cdots = P^{(k-1)}(0) = 0$;
- $\int_0^1 x^{k-1} (P^{(k)}(x))^2 dx < \infty$.

Soundararajan, Bull. AMS, 2006

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In particular, GPY reaches bounded gaps iff $S(k, P) > 4/k$.

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Find the maximum (supremum) $S(k) := \sup S(k;P)$ *– and suitable maximizing (essentially maximizing) weight function(s)* P.

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Proposition (Soundararajan)

The problem has a bounded solution: $S(k) < \frac{4}{k+\log_2 k - 5}$

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Solving Soundararajan's extremal problem

Soundararajan's extremal problem of $S(k)$–basically, a problem of the calculus of variation–was analysed by Conrey (unpublished, spreading around some rumour that the extremizer should be a Bessel function) and we[14], too.

[14] Farkas, B., Pintz, J., Révész, Sz. Gy., On the optimal weight function in the GoldstonPintz-Yıldırım method for finding small gaps between consecutive primes. In: *Number Theory, Analysis and Combinatorics: Proceedings of the Paul Turán Memorial Conference held August 22–26, 2011 in Budapest*, 75–104, de Gruyter, Berlin, 2013.

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Theorem

For the extremal problem (8) *we have*

$$S(k) = \lambda_1 = \frac{4(k-1)}{\alpha^2_{k-2,1}} = \frac{4}{k + 2ck^{1/3} + O(1)}. \qquad (1)$$

The only extremal functions for $S(k)$ *are nonzero constant multiples of* $q(x) := x^{-(k-2)/2} J_{k-2}(\alpha_{k-2,1}\sqrt{x})$.

[14] Farkas, B., Pintz, J., Révész, Sz. Gy., On the optimal weight function in the GoldstonPintz-Yıldırım method for finding small gaps between consecutive primes. In: *Number Theory, Analysis and Combinatorics: Proceedings of the Paul Turán Memorial Conference held August 22–26, 2011 in Budapest*, 75–104, de Gruyter, Berlin, 2013.

Maximizing the yield of the GPY method

Goldston, Pintz & Yildirim obtained already in 2008
$d_n = O(\sqrt{\log n} \log\log^2 n)$ i.o.

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Proposition

$S(k; P) > \dfrac{4}{k + Ck^{1/3}}$, with some explicit, effective $C > 0$.

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Any further improvements required further new ideas.

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The Motohashi-Pintz idea

For some $\beta > 0$ a number $n \in \mathbb{N}$ is called β-smooth iff $p|n \Rightarrow p < n^{\beta}$.

Theorem (Motohashi and Pintz, 2008)

To obtain the quasi-twin prime (bounded gap) conjecture, it suffices to prove for some $\theta > 1/2$ the following "smooth restriction" of θ-EH:
For some $b > 0$, any $A > 0$ and x large, we have

$$\sum_{q < x^{\theta},\ q\ b-\mathrm{smooth}} \max_{a} \left| \sum_{p \le x, p \equiv a \mod q} \log p - \frac{x}{\varphi(q)} \right| < C_A \frac{x}{\log^A x}.$$

Moreover, it suffices to restrict the maximum (over a) to numbers with $\Pi_k(a) := \prod_{j=1}^{n}(a + h_k) \equiv 0 \mod q$, for some admissible k-tuple $h_1, \ldots, h_k$.

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However, experts doubted if this weakening could be accessible. . .

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Bounded gaps between primes

Theorem (Y. Zhang, May 2013)

$\liminf_{n\to\infty} p_{n+1} - p_n < 70,000,000.$

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And the key to this indeed was:

Lemma (Y. ZHANG, May 2013)

For $\theta = 0.503$, any $A > 0$ and x large, we have

$$\sum_{q<x^\theta,\ q \text{ smooth}} \max_{a, \Pi_k(a)\equiv 0 \mod q} \left| \sum_{p\le x, p\equiv a \mod q} \log p - \frac{x}{\varphi(q)} \right| < C_A \frac{x}{\log^A x}.$$

Even more bounded prime gaps

In 2013, Tao, Green, Harcos, Sutherland, Pintz, Paldi, Elsholtz, Engelsma, Kowalsky, Hannes, Michel, Nelson, Trevino, Hou-Sun, Morrison... cooperated on a polymath project (Tao's blog) to bring down the constant 70,000,000.

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They proved $\theta = 1/2 + 7/300$ in the above smooth Bombieri-Vinogradov type Lemma of Zhang, and then $p_{n+1} - p_n < 4800$ i.o.

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But development went over of this result quickly.

And still even more bounded prime gaps

Theorem (Maynard, Nov. 2013)

It holds $p_{n+1} - p_n \leq 600$ *infinitely often. (Current best: 246.)*
Moreover, for arbitrary k *there is a bound* $C(k)$ *such that even the gaps between* k *consecutive primes are bounded by* $C(k)$*: i.e.* $p_{n+k} - p_n \leq C(k)$ ∞ *often.*

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Some related further "futuristic" results

Are there arbitrarily long AP(=Arithmetic Progression) in the twin primes?

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Theorem

If (r_n) is the sequence of Polignac numbers, then $r_{n+1} - r_n$ is bounded. Hence in particular the lower density of Polignac numbers is positive and by Szemerédi's Theorem there are arbitrarily long AP among them.

Estimations of $p_{n+1} - p_n$ from above

Related to small gaps, it is natural to ask for estimations of $p_{n+1} - p_n$ from above.

[16] Iwaniec, Henryk; Pintz, János Primes in short intervals. Monatsh. Math. 98 (1984), no. 2, 115–143.

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An overview of the work of János Pintz in Number Theory

Szilárd Gy. Révész

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Is there always a prime between two squares? In $[x, x + \sqrt{x}]$*?*

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Then in 2001 with R. Baker and Harman[17] they proved that $[x, x + x^{0.525}]$ contains a prime.

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Again, their result could not be improved upon since then.

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All odd natural numbers are the sum of three primes.

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The "2-Goldbach" problem is even harder.

It is known that most even numbers satisfy Goldbach's supposition, but there can be "Goldbach exceptional" numbers, whose number up to x is denoted by $E(x)$.

Approximations to the Goldbach Problem

Theorem (Pintz, 2018)

[18] *Exceptional numbers are at most* $E(x) \le x^{0.72}$ *in number.*

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One can also ask if there is a two-Goldbach number in any short interval $[x, x + x^\alpha]$. Conjecturally, x^α should just be 2, but we are quite far from anything close to it.

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From the estimate for $E(x)$ it trivially follows that $\alpha = 0.72$ is good.

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...and then Baker, Harman, and Pintz[19] reached $\alpha = 0.0335$. Again, this record stands ever since.

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János wrote some twenty papers on related questions, achieving very sharp, in many sense ultimate results.

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Littlewood's Problem

In 1939 Littlewood[20] posed the problem of finding explicit, effective oscillation estimates for the remainder term in the PNT, given the existence of a ζ-zero.

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According to the Riemann-von Mangoldt formula,

$$\Delta(x) = \sum_{\rho,\ |\gamma|<x} \frac{x^\rho}{\rho} + O(\log^2 x). \qquad (2)$$

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So, one might expect an oscillation about as large as $|x^\rho/\rho| = x^{\beta_0}/|\rho_0|$, this term occurring in the sum.

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In fact, in the theory of entire functions analogous questions are well explored.

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Characteristic to the behavior of f are its maximum modulus, its minimum modulus, the number of its zeroes, its absolute value sum maximum and even its largest term size:

- $M(r) := \max_{|z|=r} |f(z)|$.

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Obviously $L(r), M(r) \leq Q(r)$, but in complex variables it is clarified that these characteristics are closely related, in particular for entire functions of finite order.

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We can as well ask for results telling that also $Q(r)$ is not much larger than $M(r)$, e.g.

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Example

Take, e.g., $f(z) = e^z$.
Then $M(r) = Q(r) = e^r$, but also $L(r) = \frac{1}{[r]!} r^{[r]} \approx \frac{c}{\sqrt{n}} e^r$,
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- The Riemann-von Mangoldt sum is not a Taylor series of an entire function.
- Even if we substitute $u = \log x$, even in u it becomes only a (complex) exponential sum.
- Instead of being convergent, the series diverges for all x.
- The largest term occurs later and later, so that working with the truncated partial sum ($|\rho| \le x$), for larger X the largest term may already be out of the range for x.

Littlewood's Problem solved

It was Turán, who essentially solved the problem of Littlewood. Using his Power-Sum Theory, he obtained (somewhat better than) $|\Delta(x)| \gg x^{\beta_0 - \varepsilon}$ with effective bounds.

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Also Pintz could prove good localizations: e.g. in all intervals $[x, x^{1+\varepsilon}]$ almost as large oscillations occur.

Oscillation and sign changes

Once we know that for $x > x_0$ we have in $[x, x^{1+\varepsilon}]$ some large positive and some large absolute value negative values as well, we immediately see that $\pi(x) - \mathrm{li}\,(x))$ changes sign infinitely often; in fact, at least as many times as $\approx \frac{1}{\varepsilon} \log\log x$.

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These proofs are similar, yet from one form one cannot always derive the other–seemingly small differences cause technical obstacles, thus separate proofs are needed.

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Further records about oscillation results

At the end of his series on these oscillation results Pintz proved $V(x) \gg \log x/(\log\log x)^3$ effectively.

[21] As an analyst, I would remark that $\omega(x)$ can be expressed in a form of the well-known "Legendre transform" of $\eta(t)$. The meaning of $xe^{-\omega(x)}$ is just the largest term in the Riemann-von Mangoldt sum.

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Ingham initiated this direction with taking a function $\eta(t) \to 0$ and assuming $\zeta(s) \neq 0$ in $\Re s \geq 1 - \eta(t)$. His result was that then

$$|\Delta(x)| \leq x \exp(-\frac{1}{2}\omega_\eta(x)),$$

where $\omega_\eta(x) := \min_{t\geq 1}(1-\eta(t))\log x + \log t$ is the natural "conjugate function"[21] to $\eta(x)$.

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Sharp results

János improved this to $x\exp(-(1-\varepsilon)\omega_\eta(x))$, a result essentially sharp.

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Sharp, because, conversely, he proved:

Theorem (1980)

If there are infinitely many zeroes $\rho_k = \beta_k + i\gamma_k$ with $\beta_k > 1 - \eta(\gamma_k)$, then we have

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Theorem (2018)

Moreover, even
$D(x) := \frac{1}{X}\int_1^X |\Delta(x)|dx \geq x(\exp(-(1-\varepsilon)\omega_\eta(x)).$

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With J. Kaczorowski Pintz also extended investigations to other, more general situations, e.g. essentially optimal lower bounds for mean values of a wide class of arithmetical

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Let us concentrate on the fundamental result that $|\Delta(x)| \geq (1-\varepsilon)\frac{x^{\beta_0}}{|\rho_0|}$. Was it an easy kill? After all, the term is there, the sum has only slightly more terms than x, why the whole contribution could not be extracted?

WARNING ! Actually, there are at least two terms of that size, because with ρ_0 also $\overline{\rho_0}$ appears. But in a very precise mathematical sense (e.g. posing the question in the class of Beurling prime number distribution), it can be shown that proving $(2-\varepsilon)\frac{x^{\beta_0}}{|\rho_0|}$ is impossible. (The best constant is $\pi/2$.) That is, the interference, what Littlewood was so concerned about, does indeed occur!.

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Indeed key to the proofs was a masterly used kernel averaging. Pintz' choice of the kernel was $K(s) := e^{ks^2+ms}$.

Alexandar Ivic on Pintz' proofs

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MR820251 (86m:11071) 11N05 11N30
Pintz, J. [Pintz, János] (H-AOS)

★**Oscillatory properties of the remainder term of the prime number formula.**
Studies in pure mathematics, 551–560, *Birkhäuser*, *Basel*, 1983.

This is one of a series of papers by the author on the remainder term in the prime number formula [Acta Arith. **36** (1980), no. 4, 341–365; ibid. **37** (1980), 209–220; Studia Sci. Math. Hungar. **12** (1977), no. 3-4, 345–369; ibid. **13** (1978), no. 1-2, 29–42; ibid. **15** (1980), no. 1-3, 215–230; MR0585891-f]. In the present paper the author considers $\Delta(x) = \psi(x) - x = \sum_{p^a \leq x} \log p - x$ and proves two theorems on the oscillation of $\Delta(x)$. The main result is the following: Let $\rho_1 = \beta_1 + i\gamma_1$ be a zero of $\zeta(s)$ such that $\beta_1 \geq \frac{1}{2}$, $\gamma_1 > 0$. Then for $T \geq \max(\gamma_1^{400}, c_1)$ there exists $x \in [T^{1/4}, T]$ such that $|\Delta(x)| > c_2 x^{\beta_1} \gamma_1^{-50}$, where c_1, c_2 denote absolute, positive constants. The proof is based on the shrewd use of the integral $\int_{-\infty}^{\infty} \exp(At - Bt^2)\,dt = (\pi/B)^{1/2} \exp(A^2/4B)$ $(\mathrm{Re}\, B > 0)$ to single out the effect of ρ_1 on the behavior of $\Delta(x)$, and then on P. Turán'spower-sum method [see Turán, *On a new method of analysis and its applications*, Wiley, New York, 1984; MR0749389] to bound from below the relevant sum over zeta-zeros. Along with the aforementioned papers on the same subject (parts I and II contain proofs of Theorems 1 and 2 stated in the present paper) the author makes a significant contribution towards the elucidation of the behavior of the error terms in the prime number theorem.

{For the collection containing this paper see MR0820203}

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In any case, I went even with some of my friends, including János, too. The largest auditorium was fully packed. The Board of the Kandó Sportsclub, sitting at the large catedra, might have not expected such a crowd.

János to speak publicly.

The assembly, like such meetings in general at that time, was terribly boring. This was the era of Brezhnev and Kádár, so except for a few insiders, everyone else was supposed to listen to the speakers only (and possibly applaud ...).

[22] Which, of course, we couldn't see at the time, because it was banned, locked in the box and kept secret for another decade.

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They asked for his name, and if I remember correctly he had to write it on a piece of paper and an assistant organizer took it to the chairman's table. There were surreal moments, like in Péter Bacsó's satirical film, "The Witness"[22].

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One of the organizers, clearly audibly through the microphone, reported to the president with shock:

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Let's see a little taste of the film archive shown to us...